B. Sc. Part III (Diff. Eqns) 7th Paper

$\boxed{\text{SDE VC}}$

Q. Solve

$$x^2 \frac{d^2y}{dx^2} + x\frac{dy}{dx} - y = 0,$$

given that $x + \frac{1}{x}$ is one integral.

Soln The given equation

$$x^2 \frac{d^2y}{dx^2} + x\frac{dy}{dx} - y = 0$$

$$\Rightarrow \frac{d^2y}{dx^2} + \frac{1}{x}\frac{dy}{dx} - \frac{y}{x^2} = 0 \quad \text{---(1)}$$

comparing it with $\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = R$,

we have

$$P = \frac{1}{x},\ Q = -\frac{1}{x^2},\ R = 0 \quad \text{---(2)}$$

Given that $x + \frac{1}{x}$ is one integral of (1).

Let $u = x + \frac{1}{x} = \frac{x^2+1}{x}$

$\therefore u$ is a part of CF of the solution of (1).

Let $y = uv$ be the complete solution of (1).

Then,

$$\frac{d^2v}{dx^2} + \left(P + \frac{2}{u}\frac{du}{dx}\right)\frac{dv}{dx} = \frac{R}{u}$$

or, $$\frac{d^2v}{dx^2} + \left[\frac{1}{x} + \frac{2x}{x^2+1}\frac{d}{dx}\left(x + \frac{1}{x}\right)\right]\frac{dv}{dx} = 0$$

$$\Rightarrow \frac{d^2v}{dx^2} + \left[\frac{1}{x} + \frac{2x}{x^2+1}\cdot\left(1-\frac{1}{x^2}\right)\right]\frac{dv}{dx} = 0$$

$$\Rightarrow \frac{d^2v}{dx^2} + \left[\frac{1}{x} + \frac{2x}{x^2+1}\cdot\left(\frac{x^2-1}{x^2}\right)\right]\frac{dv}{dx} = 0$$

$$\Rightarrow \frac{d^2v}{dx^2} + \left[\frac{1}{x} + \frac{2(x^2-1)}{x(x^2+1)}\right]\frac{dv}{dx} = 0$$

$$\Rightarrow \frac{d^2v}{dx^2} + \left[\frac{x^2+1+2(x^2-1)}{x(x^2+1)}\right]\frac{dv}{dx} = 0$$

$$\Rightarrow \frac{d^2v}{dx^2} + \frac{3x^2-1}{x(x^2+1)}\frac{dv}{dx} = 0$$

$$\Rightarrow \frac{d}{dx}\left(\frac{dv}{dx}\right) + \frac{3x^2-1}{x(x^2+1)}\frac{dv}{dx} = 0 \quad \text{———} (3)$$

$$\text{put } \frac{dv}{dx} = q \quad \text{———} (4)$$

$$\Rightarrow \frac{dq}{dx} + \frac{3x^2-1}{x(x^2+1)}\, q = 0$$

$$\Rightarrow \frac{dq}{q} + \frac{3x^2-1}{x(x^2+1)}\, dx = 0$$

$$\Rightarrow \frac{dq}{q} + \frac{4x^2-(x^2+1)}{x(x^2+1)}\, dx = 0$$

$$\Rightarrow \frac{dq}{q} + \left[\frac{4x}{x^2+1} - \frac{1}{x}\right] dx = 0$$

$$\Rightarrow \frac{dq}{q} + \frac{4x\,dx}{x^2+1} - \frac{dx}{x} = 0$$

Integrating, we get

$$\Rightarrow \int \frac{dq}{q} + 2\int \frac{2x\,dx}{x^2+1} - \int \frac{dx}{x} = 0$$

$$\Rightarrow \log q + 2\log(x^2+1) - \log x = \log K$$

$$\Rightarrow \log \frac{q(x^2+1)^2}{x} = \log K$$

$$\Rightarrow K = \frac{q(x^2+1)^2}{x}$$

$$\Rightarrow q = \frac{Kx}{(x^2+1)^2}$$ But $q = \frac{dv}{dx}$ from (4)

$$\Rightarrow \frac{dv}{dx} = \frac{Kx}{(x^2+1)^2} \Rightarrow dv = \frac{Kx\,dx}{(x^2+1)^2}$$

Integrating, we get

$$\int dv = K\int \frac{x\,dx}{(x^2+1)^2} \quad \text{———(5)}$$

Put $x^2+1 = z$. Differentiating, we get

$$\Rightarrow 2x\,dx = dz$$

$$\Rightarrow x\,dx = \frac{dz}{2}$$

From (5)

$$\int dv = k \int \frac{dz}{2z^2}$$

$$\Rightarrow v = \frac{k}{2} \cdot x \frac{-1}{z} + k_1$$

But $z = x^2 + 1$

$$\Rightarrow v = \frac{-k}{2(x^2+1)} + k_1 \quad \text{——— (6)}$$

$\because$ $y = uv$ is the complete solution

and $u = x + \frac{1}{x} = \frac{x^2+1}{x}$

$$v = \frac{-k}{2(x^2+1)} + k_1$$

$$\Rightarrow y = uv = \frac{x^2+1}{x} \cdot \left[\frac{-k}{2(x^2+1)} + k_1 \right]$$

$$\Rightarrow y = \frac{-k}{2x} + \frac{k_1}{x}(x^2+1)$$

Put $-\frac{k}{2} = k_2$

$$\Rightarrow y = \frac{k_2}{x} + k_1\left(x + \frac{1}{x}\right).$$

is the complete solution.